INFLUENCE OF THE HALL EFFECT ON HYPERSONIC FLOW PAST A WEDGE

L. A. Panchenko

Zhurnal prikladnoi mekhaniki i tekhnicheskoi fiziki, No. 1, pp. 21-26, 1965

This paper deals with the problem of the steady-state hypersonic flow of an inviscid compressible gas past a wedge. Inside the wedge a magnetic field is excited in a direction perpendicular to the generator. The flow in the region of perturbation is investigated on the basis of the ordinary equations of magnetohydrodynamics and Ohm's law, written for the case where the Hall effect is taken into account. The system of equations obtained has been solved numerically on a computer by the method of finite differences. The results show that for the given problem the Hall effect intensifies the magnetohydrodynamic action of the magnetic field on the flow. M. D. Ladyzhenskii [1] has also studied hypersonic flow past bodies from inside which a magnetic field is excited. He has investigated the influence of a strong magnetic field on the flow for the case where the Hall effect is neglected. The object of the present study is to determine the importance of the Hall effect.

Suppose a wedge is exposed to the steady-state hypersonic flow of an inviscid compressible gas. Inside the wedge a magnetic field is excited. The field vector is perpendicular to the generator of the wedge and has a constant modulus H^*. Electrical conduction in the undisturbed flow is neglected. In the region beyond the shock wave the equations of magnetohydrodynamics hold true, and, since we are considering the case in which $\omega\tau \neq 0$, where ω is the gyrofrequency of the electrons in the magnetic field, and τ is the average time between electron-ion collisions, we shall take Ohm's law in the form [2]:

$$\mathbf{j} = \sigma_0\left(\mathbf{E} + \frac{1}{c}\mathbf{V}\times\mathbf{H}\right) - \frac{\omega\tau}{H}\mathbf{j}\times\mathbf{H} \tag{1}$$

where σ_0 is the conductivity in the absence of a magnetic field (assumed constant); H, E are the magnetic and electric field vectors, respectively; c is the speed of light in a vacuum; V is the gas velocity; and j is the current density vector.

We shall relate the velocity in the region of perturbation to the velocity of the undisturbed flow V_∞, the intensity of the magnetic field to the quantity H^*, the density to the density at infinity ρ_∞, the current density to the quantity $\sigma_0 V_\infty H^*/c$, the electric field intensity to the quantity $V_\infty H^*/c$, the pressure to twice the velocity head in the undisturbed flow $\rho_\infty V_\infty^2$, and the space coordinates to the length of the generator of the wedge L. In dimensionless variables the equations of magnetohydrodynamics and Ohm's law may be written thus:

$$\begin{gathered}(\mathbf{V}\cdot\nabla)\mathbf{V} = -\frac{1}{\rho}\nabla p + \frac{q}{\rho}\mathbf{j}\times\mathbf{H}, \qquad \operatorname{div}\rho\mathbf{V} = 0, \qquad q = \frac{\sigma_0 H^* L}{c^2\rho_\infty V_\infty}, \\ \frac{1}{q}\rho\mathbf{V}\cdot\nabla\left(\frac{V^2}{2} + \frac{1}{\varkappa-1}\frac{p}{\rho}\right) = \mathbf{E}\cdot\mathbf{j}, \qquad \mathbf{j} = \frac{1}{R_m}\operatorname{rot}\mathbf{H}, \qquad R_m = \frac{4\pi\sigma_0 L V_\infty}{c^2}, \\ \mathbf{j} = \mathbf{E} + \mathbf{V}\times\mathbf{H} - \frac{\omega\tau}{H}\mathbf{j}\times\mathbf{H}, \qquad \operatorname{rot}\mathbf{E} = 0, \qquad \operatorname{div}\mathbf{H} = 0,\end{gathered} \tag{2}$$

where V, H, ρ, and p are, respectively, the velocity vector, magnetic field strength vector, density, and pressure, all in dimensionless form, and $\varkappa$ is the ratio of specific heats.

Equations (2) contain the dimensionless quantities q and R_m. Their order of magnitude was estimated in [1]. Here we shall consider the cases $R_m \sim 1$, $q \sim 1/\varepsilon$, where $\varepsilon = (\varkappa - 1)/(\varkappa + 1)$ is the ratio of the density in front of the shock wave.

To equations (2) we must add the conditions at the shock wave and the boundary conditions at the body and at infinity. In dimensionless variables, assuming that the Mach number at infinity is infinitely large, the conditions at the shock wave assume the form:

$$p_2 = \frac{2}{\varkappa+1}\sin^2\beta, \quad \rho_2 = \frac{\varkappa+1}{\varkappa-1}, \quad V_{2\tau} = \cos\beta, \quad V_{2n} = \frac{\varkappa-1}{\varkappa+1}\sin^2\beta, \quad \mathbf{H}_2 = \mathbf{H}_1, \tag{3}$$

where the subscript 2 relates to the region beyond the shock wave, τ and n are, respectively, unit vectors of the tangent and normal to the shock wave, and β is the local angle of inclination of the discontinuity with respect to the direction of the velocity in the undisturbed flow. At the body we have the no-flow condition $V_n = 0$. In the undisturbed flow the following equations hold for the magnetic field:

$$\operatorname{div}\mathbf{H}=0,\ \operatorname{rot}\mathbf{H}=0\,.$$

Equations (2) and (3) constitute the complete system of equations of the problem. The shape of the shock wave is determined in the process of its solution.

We project Eqs. (2) onto the axes of the Cartesian coordinate system x, y, z, which we shall select as follows: the origin is located at the nose of the wedge, the x axis is directed along the generator, and the y axis at right angles to it. By virtue of the formulation of the problem, all the functions in question are functions only of x and y and do not depend on z. The components of V along the axes x, y, z will be denoted by u, v, and w, respectively. We then get:

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{q}{\rho}(j_y h_z-j_z h_y) \tag{4}$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\frac{q}{\rho}(j_z h_x-j_x h_z) \tag{5}$$

$$u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}=\frac{q}{\rho}(j_x h_y-j_y h_x) \tag{6}$$

$$\frac{\partial\rho u}{\partial x}+\frac{\partial\rho v}{\partial y}=0,\qquad \frac{\partial h_x}{\partial x}+\frac{\partial h_y}{\partial y}=0 \tag{7}$$

$$j_x=\frac{1}{R_m}\frac{\partial h_z}{\partial y},\qquad j_y=-\frac{1}{R_m}\frac{\partial h_z}{\partial x},\qquad j_z=\frac{1}{R_m}\left(\frac{\partial h_y}{\partial x}-\frac{\partial h_x}{\partial y}\right) \tag{8}$$

$$\begin{aligned}
j_x&=\frac{1}{1+\omega^2\tau^2}\Big[\Big(1+\frac{\omega^2\tau^2}{h^2}h_x^2\Big)(E_x+vh_z-wh_y)+\\
&+\Big(\frac{\omega^2\tau^2}{h^2}h_xh_y-\frac{\omega\tau}{h}h_z\Big)(E_y+wh_x-uh_z)+\Big(\frac{\omega^2\tau^2}{h^2}h_xh_z+\frac{\omega\tau}{h}h_y\Big)(uh_y-vh_x)\Big]\\
j_y&=\frac{1}{1+\omega^2\tau^2}\Big[\Big(\frac{\omega^2\tau^2}{h^2}h_xh_y+\frac{\omega\tau}{h}h_z\Big)(E_x+vh_z-wh_y)+\\
&+\Big(1+\frac{\omega^2\tau^2}{h^2}h_y^2\Big)(E_y+wh_x-uh_z)+\Big(\frac{\omega^2\tau^2}{h^2}h_yh_z-\frac{\omega\tau}{h}h_x\Big)(uh_y-vh_x)\Big]\\
j_z&=\frac{1}{1+\omega^2\tau^2}\Big[\Big(\frac{\omega^2\tau^2}{h^2}h_xh_z-\frac{\omega\tau}{h}h_y\Big)(E_x+vh_z-wh_y)+\\
&+\Big(\frac{\omega^2\tau^2}{h^2}h_yh_z-\frac{\omega\tau}{h}h_x\Big)(E_y+wh_x-uh_z)+\Big(1+\frac{\omega^2\tau^2}{h^2}h_z^2\Big)(uh_y-vh_x)\Big]
\end{aligned} \tag{9}$$

$$\begin{aligned}
&\frac{\rho}{q}\left[u\frac{\partial}{\partial x}\left(\frac{u^2}{2}+\frac{v^2}{2}+\frac{w^2}{2}+\frac{1}{\varkappa-1}\frac{p}{\rho}\right)\right]+\\
&+\frac{\rho}{q}\left[v\frac{\partial}{\partial y}\left(\frac{u^2}{2}+\frac{v^2}{2}+\frac{w^2}{2}+\frac{1}{\varkappa-1}\frac{p}{\rho}\right)\right]=E_xj_x+E_yj_y+E_zj_z
\end{aligned} \tag{10}$$

$$\frac{\partial E_z}{\partial y}=0,\qquad \frac{\partial E_z}{\partial x}=0,\qquad \frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}=0\,. \tag{11}$$

From Eqs. (11) we find that $E_z = 0$, since it is assumed that there is no external electric field.

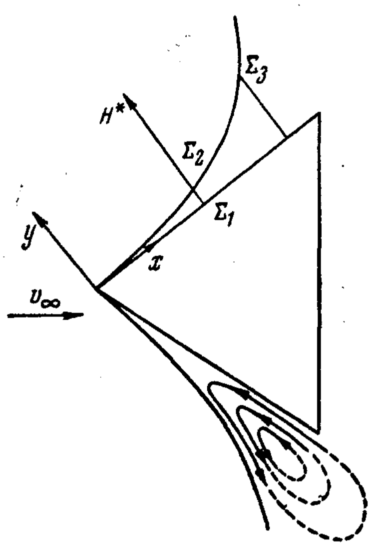

Fig. 1

From the equations of magnetohydrodynamics it follows that the equation div j = 0 is everywhere fulfilled or, in accordance with Gauss' theorem,

$$\int_\Sigma j_n d\Sigma=0$$

where Σ is some closed surface, and j_n is the current density component normal to the surface. Since in our problem nothing depends on z, we shall take as the surface Σ a piece of the surface (the intersection of which with the plane z = 0 is shown in Fig. 1) lying between the planes z = 0 and z = 1. $\Sigma = \Sigma_1 + \Sigma_2 + \Sigma_3$.

Note that at the surface of the wedge $j_n = 0$, as follows from the formulation of the problem. In front of the shock wave the gas is nonconducting, i. e., there are no currents from the shock wave; therefore we have, in turn:

$$\int_{\Sigma_1} j_n\,d\Sigma=0,\quad \int_{\Sigma_2} j_n\,d\Sigma=0,\quad \int_{\Sigma_3} j_n\,d\Sigma=0\quad\text{or}\quad \int_0^Y j_x\,dy=0\,. \tag{12}$$

Thus, from the formulation of the problem it follows that no electric current flows in the direction of the x axis.

Assuming that the parameter $\varepsilon = (\varkappa - 1)/(\varkappa + 1)$ is small, we shall estimate the quantities entering into equations (4)-(11) for the case $\omega\tau \sim 1$. By analogy with [1], we have:

$$x \sim 1, \quad y \sim \varepsilon, \quad u \sim 1, \quad v \sim \varepsilon, \quad \rho \sim 1/\varepsilon \ ,$$
$$\partial p / \partial x \sim \varepsilon, \quad p = \sin^2\theta, \quad h_y = 1 + O(\varepsilon), \quad j_z \sim 1 \ .$$

Further, from Eqs. (5), (8), (6), (9), and (10), (11) we have, respectively:

$$h_x \sim \varepsilon, \quad j_x \sim 1, \quad h_z \sim \varepsilon, \quad j_y \sim \varepsilon, \quad w \sim 1 \ ,$$
$$E_x \sim 1, \quad E_y \sim \varepsilon, \quad E_x = f(x) + \varepsilon\varphi(y), \quad f(x) \sim 1, \quad \varphi(y) \sim 1 \ . \tag{13}$$

All these estimates, obtained for the case $\omega\tau \sim 1$, are confirmed by the solution.

Neglecting in Eqs. (4)-(11) terms of the order of ε in comparison with 1 and using (12), we obtain:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{q}{\rho}j_z, \qquad u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} = \frac{q}{\rho}j_x \ ,$$
$$u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} = \frac{2q\varepsilon}{p}\rho\left[wj_x - uj_z - E_x j_x\right] \ , \tag{14}$$
$$j_x = \frac{E_x - w + \omega\tau u}{1 + \omega^2\tau^2}, \qquad j_z = \frac{\omega\tau w - \omega\tau E_x + u}{1 + \omega^2\tau^2}, \qquad E_x = \frac{1}{Y}\int_0^Y (w - \omega\tau u)\, dy \ ,$$

where $Y = Y(x)$ is the equation of the shock wave. The last equation in system (14) is obtained from Eqs. (12), (13) and the equation for j_x of system (14).

At the shock wave from (3) we have

$$p_2 = \sin^2\theta, \quad \rho_2 = 1/\varepsilon, \quad u_2 = \cos\theta$$
$$v_2 = (dY/dx)\cos\theta - \varepsilon\sin\theta, \quad w = 0 \qquad \text{for} \quad Y = Y(x) \tag{15}$$

where θ is the half-angle of the wedge. We shall go over to the new variables x and ψ, determining ψ as follows:

$$\frac{\partial\psi}{\partial x} = \rho v, \qquad \frac{\partial\psi}{\partial y} = -\rho u, \qquad \psi = 0 \ \text{(at the surface of the wedge)}.$$

Then from (14) we get

$$u\frac{\partial u}{\partial x} = -\frac{q}{\rho}j_z, \qquad u\frac{\partial w}{\partial x} = \frac{q}{\rho}j_x \ ,$$
$$\frac{\partial \rho}{\partial x} = \frac{2q\varepsilon}{p}\rho\left[\frac{w}{u}j_x - j_z - \frac{E_x}{u}j_x\right], \qquad E_x = -\frac{1}{Y}\int_0^Y (w - \omega\tau u)\frac{d\psi}{\rho u} \ ,$$
$$j_x = \frac{E_x - w + \omega\tau u}{1 + \omega^2\tau^2}, \qquad j_z = \frac{\omega\tau w - \omega\tau E_x + u}{1 + \omega^2\tau^2} \ . \tag{16}$$

System (16) describes the gas flow in the region of perturbation beyond the shock wave and determines the electric currents and the electric field in this region. We shall find the shape of the shock wave. From the continuity equation in the variables x and ψ we have:

$$\frac{\partial(v/u)}{\partial\psi} + \frac{\partial}{\partial x}\frac{1}{\rho u} = 0, \qquad v = -u\int_0^\psi \frac{\partial}{\partial x}\frac{1}{\rho u}\, d\psi \ . \tag{17}$$

From boundary conditions (15) we have:

$$v_2 = \frac{dY}{dx}\cos\theta - \varepsilon\sin\theta \ . \tag{18}$$

Using the expression $\psi = -x\sin\theta - y\cos\theta$ for the current function in the undisturbed flow, taking continuity at the shock wave into account, we get:

$$\Psi = -x\sin\theta - Y\cos\theta. \tag{19}$$

From (18) and (19) we find:

$$v_2 = -\frac{d\Psi}{dx} - (1+\varepsilon)\sin\theta \ . \tag{20}$$

Substituting (20) in (17), we find the equation for determining the shape of the shock wave in the new variables

$$\frac{d\Psi}{dx} = -\sin\theta(1+\varepsilon) + \cos\theta\int_0^{\psi}\frac{\partial}{\partial x}\frac{1}{\rho u}\,d\psi \ , \tag{21}$$

from which, using (19), we can find Y(x).

System of equations (16) and equation (21) were computed numerically on an electronic computer by the method of finite differences. Solutions were obtained for $\omega\tau$ = 0.1, 0.5, 1, 10, 100. In the computations it was assumed that θ = 40°, $\varkappa$ = 1.4. The correctness of the computations was checked by seeing whether for $\omega\tau$ = 0 the solution coincided with the results of [1]. The integration step Δx = 0.01.

Upon halving the step, i.e., for Δx = 0.005, we found changes in the fourth to fifth places of the quantities computed, i.e., Δx = 0.01 gave sufficient accuracy.

We shall estimate the order of the quantities entering into (21) with respect to $\omega\tau$. From (21) we have

$$E_x \sim \omega\tau, \quad j_x \sim \omega\tau, \quad w \sim \omega\tau \quad \text{for} \quad \omega\tau \ll 1 \ .$$

The order of the remaining quantities does not change and is given by the estimates made for $\omega\tau \sim 1$. Thus, in the limit as $\omega\tau \to 0$ we have the solution obtained in [1]. Now let $\omega\tau \gg 1$, then

$$w \sim \frac{1}{\omega\tau}, \quad j_x \sim \frac{1}{\omega\tau}, \quad E_x \sim \omega\tau \ .$$

The order of the remaining quantities does not change and is given by the estimates made for the case $\omega\tau \sim 1$. When $\omega\tau \gg 1$ the equations of system (21) assume a simpler form.

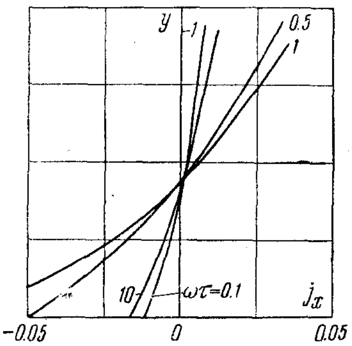

Fig. 2

We shall put $E_x = \omega\tau E_x'$, where $E_x' \sim 1$. Then

$$u\frac{\partial u}{\partial x} = -\frac{a}{\rho}j_z, \qquad \frac{\partial\rho}{\partial x} = -\frac{2q\varepsilon}{p}\rho\frac{E_x'^2}{u}$$
$$j_z = -E_x', \qquad E_x' = \frac{1}{Y}\int_0^Y\frac{d\psi}{\rho}, \quad j_x = 0, \qquad w = 0 \ . \tag{22}$$

When $\omega\tau \to \infty$ system (16) has a limit solution satisfying system (22).

If the electric field $E^* = E + V \times H$ is perpendicular to the magnetic field (in our case this condition is fulfilled), then from Eq. (1) we can obtain [2]:

$$\mathbf{j} = \frac{1}{1+\omega^2\tau^2}\left(\mathbf{E}^* - \frac{\omega\tau}{H}\mathbf{E}^* \times \mathbf{H}\right) \tag{23}$$

in the dimensionless variables introduced above. In (23) the second term on the right is the current perpendicular both to E^* and to H, the so-called Hall current. The projections of this current j^* on the coordinate axes will be

$$\mathbf{j}^* = \frac{\omega\tau}{1+\omega^2\tau^2}\frac{\mathbf{H}\times\mathbf{E}^*}{H}, \quad j_x^* = \frac{\omega\tau u}{1+\omega^2\tau^2}, \quad j_y^* = 0, \quad j_z^* = \frac{\omega\tau}{1+\omega^2\tau^2}[w - E_x] \ .$$

Near the wall of the wedge, where the velocity u is small, the component of the Hall current along the x axis will be small, and the main contribution to j_x will be made by the component of the direct current along the x axis, i.e., the current parallel to E^*, as may be seen from Eq. (23). As we move away from the wall of the wedge, the velocity component u increases: the components of the Hall current and the direct current along the x axis have the same sign. The projection of the total current on the x axis with respect to a section x = const passes through zero and changes sign (as confirmed by the numerical results presented in Fig. 2 in the form of graphs of the function $j_x = j_x(y)$ for x = 0.4 and different $\omega\tau$); therefore there is no total current in the direction of the x axis. On the basis of the Hall current we also obtained the estimates:

$$j_x \sim 1/\omega\tau, \quad j_z \sim 1 \quad \text{for} \quad \omega\tau \gg 1,$$
$$j_x \sim 1, \quad j_z \sim 1 \quad \text{for} \quad \omega\tau \sim 1,$$
$$j_x \sim \omega\tau, \quad j_z \sim 1 \quad \text{for} \quad \omega\tau \ll 1.$$

The results obtained confirm the estimates of orders of magnitude in relation to $\omega\tau$.

Figures 3 and 4 give the numerical results for $j_x(x)$ and $w(x)$, respectively, for $y = 0$ and different $\omega\tau$. These show that the transverse velocity component w, appearing when $\omega\tau \neq 0$, increases with increase in $\omega\tau$, reaches a maximum, and then starts to decline. Even at $\omega\tau = 100$ the values of j_x and w are negligibly small.

No electric current flows in the direction of the x axis, therefore we get an electric field $E_x = E_x(x)$. In Fig. 4 the broken line shows the dependence of $0.1E_x/\omega\tau$ on x for different $\omega\tau$, which confirms the conclusion that $E_x \sim \omega\tau$.

In [1] it was shown that when a magnetic field is excited inside bodies placed in a hypersonic flow the flow may separate from the body. This is attributable to the action of ponderomotive forces in a direction opposed to the motion. The region beyond the separation point is the separation zone of vortex flow.

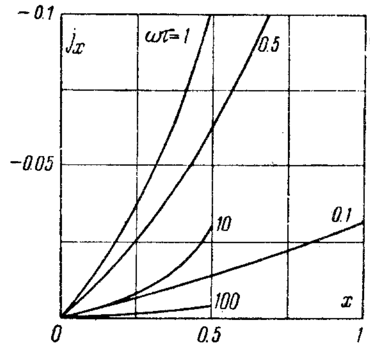

Fig. 3.

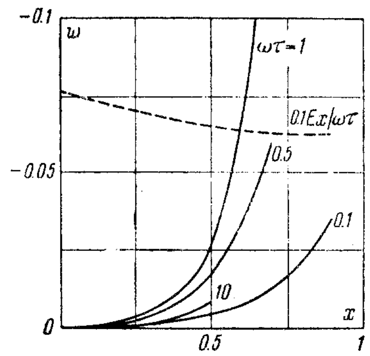

Fig. 4

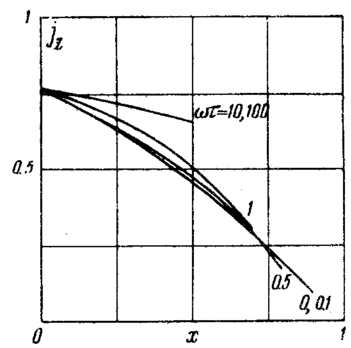

Fig. 5

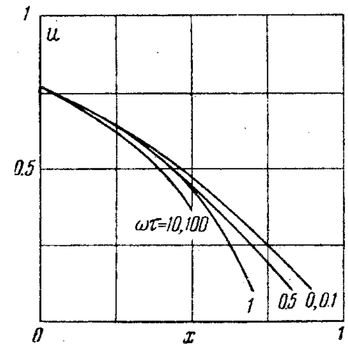

Fig. 6

Calculations for the case $\omega\tau \neq 0$ have shown that with increase in $\omega\tau$ the coordinate x_* of the separation point is displaced upwards with respect to the flow. With increase in $\omega\tau$ the quantity j_z increases, as may be seen from the equation for j_z of system (16) and confirmed by the numerical results for $y = 0$ presented in Fig. 5. The increasing ponderomotive force causes more rapid deceleration of the flow. Figure 6 shows the numerical results for u at the wall of the wedge in relation to various $\omega\tau$. Here are several values of x_* as a function of $\omega\tau$ for $\varkappa = 1.4$:

$$\omega\tau = 0, \quad 0.1, \quad 0.5, \quad 1, \quad 10, \quad 100,$$
$$x_* = 1, \quad 0.995, \quad 0.855, \quad 0.72, \quad 0.59, \quad 0.59.$$

The results of the computations confirmed the conclusions made on the basis of an analysis of system (16), that for large $\omega\tau$ the flow in the region of perturbation tends to a certain limiting flow. This is shown by calculations for $\omega\tau = 10$ and $\omega\tau = 100$.

Figure 1 shows the pattern formed by the lines of electric current. There is no current in the direction of the x axis, but the current lines must close somewhere. However, this region is not embraced by the computations, since the main role there is played by the component j_y, and from the estimates made for the case $\omega\tau \sim 1$ it is clear that $j_y \sim \varepsilon$.

From this study of the influence of the Hall effect on hypersonic flow past a wedge, from inside which a magnetic field is excited, we may conclude that for our problem the Hall effect intensifies the magnetohydrodynamic action of the field on the flow. This is a result of the specific features of the flow (geometry, absence of current flow in the direction of the x axis).

The author wishes to thank M. D. Ladyzhenskii for formulating the problem and discussing the progress of the work.

REFERENCES

1. M. D. Ladyzhenskii, "Hypersonic flow around bodies in magnetohydrodynamics," PMM, vol. 23, no. 6, 1959.
2. T. Cowling, Magnetohydrodynamics [Russian translation], IL, 1959

15 June 1964 Moscow